SOME THEOREMS RELATED TO THE JACOBI VARIATIONAL PRINCIPLE OF ANALYTICAL DYNAMICS*†

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Abstract

It is shown that there exists a commuting diagram of mappings between dynamics of classical systems on one side and variational principles for geodesic lines in stationary spacetimes of general relativity on the other. The construction of the mappings is based on classical Routh's and Jacobi's reduction procedures and on corresponding inverse procedures which are reviewed in the paper.

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*I am honoured that I may dedicate this work to Jerzy Plebański, a teacher and a friend of mine who a long time ago introduced me into the realm of variational principles in physics, on the occasion of his 75$^{\text{th}}$ birthday.

†This is a contribution to a volume in celebration of the 75$^{\text{th}}$ birthday of Jerzy Plebański. Due to a considerable delay of its publication, I have decided now to publish it in a preprint form.

1. Introduction

Since a long time it has been well known that the number of Newtonian differential equations of motion can be diminished by making use of the existence of some mechanical conservation laws. In the middle of the 19^{th} century a new problem of this kind was posed. If the original set of equations of motion are the Euler-Lagrange equations of a Lagrangian, and the dynamical system admits conservation laws that can be used to reduce the number of these equations, is then always possible to find a new Lagrangian such that the reduced system of differential equations can be derived as Euler-Lagrange equations of the new Lagrangian?

As is known, cf. [1], the first solution to the problem was given in 1876 by E. J. Routh who showed that when the corresponding conservation laws resulted from the occurrence of cyclic dynamical variables in the original Lagrangian, it was the Routh function that was the Lagrangian for the reduced system. If, however, the law responsible for the reduction of the system is the conservation of energy, the Routh method applied directly to the original Lagrangian does not work at all. This case required a separate treatment that was given in 1886 in a book by K. G. J. Jacobi, where a variational principle leading to differential equations satisfied by spatial trajectories in the configuration space of the dynamical system was formulated under the assumption that the original Lagrangian did not depend explicitly on the time variable. The proof of Jacobi was based on the Maupertuis principle of least action. This fact may be one of the sources of a terminological confusion which appears in many contemporary text-books on analytical dynamics, where the Jacobi principle is named Maupertuis principle, despite the fact that the latter is, for holonomic dynamical systems and for $E \neq 0$, equivalent to the Lagrange equations of the second kind which determine the motion of the system, whereas the Jacobi principle determines only the orbits of the motion.

In 1994, in [2], the present author together with P. Jaranowski have posed and solved, as they have named it, the inverse Jacobi problem: under which conditions imposed, can one restore the original motion when starting from a variational principle leading to orbits?

In [2] an attempt was also made to derive *ab initio* the standard Jacobi principle without making any use of the Maupertuis principle. During my seminar talks on the results obtained in [2], I realized that the "new" derivation of the Jacobi principle was rather complicated to convey it to the audience. In 2001, I found a very simple derivation of the Jacobi principle, published later in [3]. It makes use of the Routh method applied to an in a suitable way transformed Hamilton's action. It is so simple that it must have been undoubtedly known to some people before, although I could not find any references to it. From the point of view of methodology, this derivation is more suitable for a classroom than the traditional one, because it solves two akin problems in the same way.

The main objective of the article is to demonstrate that there exist mappings between dynamics of classical systems on one side and variational principles for geodesic lines in stationary spacetimes of general relativity on the other. The construction of the mappings is based on classical Routh's and Jacobi's reduction procedures and on corresponding inverse procedures which are proposed by the present author and P. Jaranowski in [2] and [3]. All these procedures are general theoretical methods

that belong to analytical dynamics. A review of them is presented in sections 2, 3, 4, and 5, mainly in order to fix the framework which will be employed in the next sections.

Sections 6, 7, and 8 present the classical Jacobi procedure in the working. In Sec. 6, the relation between two widely known actions for geodesics on manifolds is interpreted in terms of the Jacobi reduction of the "quadratic" action into the other one. Section 7 repeats the elementary text-book example of the Jacobi reduction of a Newtonian, holonomic Lagrangian into the Lagrangian describing orbits as geodesics in the kinetic energy metric. The action for geodesic lines in a stationary Lorentzian manifold in the coordinate time parametrization is in Sec. 8 Jacobi reduced into an action defined on the constant time hypersurface. All the examples considered in these three sections are at the end of Sec. 8 reinterpreted in terms of mappings of some of the dynamics into the other ones, and the equivalence of some of the dynamics is exhibited there.

The inverse Jacobi procedure is applied to the action considered in the previous section in Sec. 9. Its result is an action determining affinely parametrized geodesics, and the metric coefficients in this action are time independent. This fact enables one to form a composition of the Routh reduction with the inverse Jacobi procedure performed just at the beginning of the section. The outcome is a dynamics which is equivalent to the Newtonian dynamics considered in Sec. 7. This enables one to continue the discussion led at the end of the previous section and to construct two closed loops of mappings that alternatively can be considered as a commuting diagram of mappings between all the dynamics dealt with in this article. Two of the branches in this diagram may be regarded as generalizations of the correspondences between dynamics that were already discussed in the literature, cf. [4] and [5], but by methods that are rather particular, and without any reference to general principles of analytical dynamics.

In the text which follows, an abbreviated notation is used, in accordance with which expressions like e.g. $(q^i, \dot{q}^j)$ stand for sequences $(q^1, q^2, \ldots, q^n, \dot{q}^1, \ldots, \dot{q}^n)$ or, depending on ranges in which the indices vary, for some other sequences of a similar type. The summation convention is employed throughout the article.

2. Routh's theorem

Let

$$\mathcal{W}[q^\alpha] = \int_{t_1}^{t_2} \mathcal{L}\left(q^i(t), \dot{q}^\beta(t), t\right)\, dt \tag{1}$$

be an action functional describing a dynamical system in a configuration space $\mathbb{Q}^{n+1}$. The local coordinates q^α, where $\alpha, \beta = 0, 1, ..., n$, of a point in $\mathbb{Q}^{n+1}$ are functions of time, $q^\alpha = q^\alpha(t)$, called the motion of the system in $\mathbb{Q}^{n+1}$. Let further the Lagrangian $\mathcal{L}$ be non-degenerate. The form of the action (1) was written down in accordance with the assumption that $\partial\mathcal{L}/\partial q^0 = 0$, i.e. with the fact that the variable q^0 is a cyclic one.

From this assumption it follows that

$$p_0 = \frac{\partial \mathcal{L}}{\partial \dot{q}^0} := \mathcal{P}_0\left(q^i(t), \dot{q}^0(t), \dot{q}^j(t), t\right) = \text{const}, \tag{2}$$

where $i, j = 1, 2, ..., n$.

Then, cf. [1],

1. Equation (2) can be solved with respect to the variable $\dot{q}^0$ leaving us with a relation of the form
$$\dot{q}^0(t) = \phi\left(p_0, q^i(t), \dot{q}^j(t), t\right),$$
where p_0 is an arbitrary, but fixed, value of the integration constant. As a result, the variables $\left(q^0(t), \dot{q}^0(t)\right)$ can be eliminated from the system of the $n+1$ original Lagrange equations.

2. The remaining n differential equations for the variables $q^i(p_0, t)$ are again Euler-Lagrange equations of an action integral
$$\mathcal{W}_{p_0}[q^i] = \int\limits_{t_1}^{t_2} L_{p_0}\left(q^i(t), \dot{q}^j(t), t\right) dt, \tag{3}$$
where L_{p_0} is defined as
$$L_{p_0}\left(q^i, \dot{q}^j, t\right) = \mathcal{R}\left(q^i, \phi(p_0, q^k, \dot{q}^l, t), \dot{q}^j, t\right), \tag{4}$$
and where $\mathcal{R}$ denotes the Routh function
$$\mathcal{R}(q^i, \dot{q}^0, \dot{q}^j, t) := \mathcal{L}\left(q^i, \dot{q}^0, \dot{q}^j, t\right) - \dot{q}^0 p_0.$$

3. After the Euler-Lagrange equations corresponding to the action (3) have been solved for $q^i(p_0, t)$, one can find the function $q^0(p_0, t)$ by solving the differential equation
$$\dot{q}^0 = -\frac{\partial \mathcal{R}_{p_0}}{\partial p_0} = \tilde{\phi}(p_0, t), \tag{5}$$
where the function $\tilde{\phi}(p_0, t)$ is a solution of the equation
$$\mathcal{P}_0\left(q^i(p_0, t), \tilde{\phi}(p_0, t), \dot{q}^j(p_0, t), t\right) = p_0 \tag{6}$$
into which the now known functions $q^i(p_0, t)$ and $\dot{q}^j(p_0, t)$ are substituted.

3. The Routh inverse procedure

In order to determine the complete motion described by (q^0, q^i), the knowledge of a pair of functions $(L_{p_0}, \mathcal{P}_0)$, and of a constant p_0 was necessary. A natural question now arises whether this information is also sufficient to determine the functional form of the original Lagrangian $\mathcal{L}$ provided the triple $(L_{p_0}, \mathcal{P}_0, p_0)$ is known.

The answer to the question just posed is positive, and the proof proceeds as follows.

1. Suppose that a function $\mathcal{P}_0$ is given. Any Lagrangian $\mathcal{L}(q^i, \dot{q}^0, \dot{q}^j, t)$ such that
$$\frac{\partial \mathcal{L}}{\partial \dot{q}^0} = \mathcal{P}_0\left(q^i, \dot{q}^0, \dot{q}^j, t\right) \tag{7}$$

is of the form

$$\mathcal{L}(q^i, \dot{q}^0, \dot{q}^j, t) = I(q^i, \dot{q}^0, \dot{q}^j, t) + \Lambda(q^i, \dot{q}^j, t), \tag{8}$$

where

$$I(q^i, \dot{q}^0, \dot{q}^j, t) = \int \mathcal{P}_0(q^i, \dot{q}^0, \dot{q}^j, t)\, d\dot{q}^0,$$

and Λ is a quite arbitrary function of the arguments shown in (8).

2. The arbitrariness of Λ is removed by the requirement that the Routh procedure, which starts from the assumption

$$\mathcal{P}_0\left(q^i(t), \dot{q}^0(t), \dot{q}^j(t), t\right) = p_0 = \text{const}, \tag{9}$$

if applied to (7), lead to the now known Lagrangian $L_{p_0}(q^i, \dot{q}^j, t)$. As a result, one obtains

$$\Lambda(q^i, \dot{q}^j, t) = L_{p_0}(q^i, \dot{q}^j) - I\left(q^i, \varphi(p_0, q^k, \dot{q}^l, t)\, \dot{q}^j, t\right) + \varphi(p_0, q^i, \dot{q}^j, t) p_0, \tag{10}$$

where the function φ is defined in an implicit way by the equation

$$\mathcal{P}_0\left(q^i(t), \varphi, \dot{q}^j(t), t\right) = p_0. \tag{11}$$

Of course, the value of the parameter p_0 in Eqs. (9) and (11) must agree with that entering the known Lagrangian $L_{p_0}(q^i, \dot{q}^j)$.

3. The final functional form of $\mathcal{L}$, obtained in consequence of substituting Eq. (10) into Eq. (8), is

$$\mathcal{L}(q^i, \dot{q}^0, \dot{q}^j, t) = L_{p_0}(q^i, \dot{q}^j, t) + \varphi(p_0, q^i, \dot{q}^j, t)\, p_0 + \int\limits_{\varphi(p_0, q^i, \dot{q}^j, t)}^{\dot{q}^0} \mathcal{P}_0\left(q^i, \kappa, \dot{q}^j, t\right)\, d\kappa. \tag{12}$$

Depending on the number of solutions for φ admitted by Eq. (11), the solution (12) of the inverse problem may not be a unique one.

4. The Jacobi principle

Let us consider now an action functional of the form

$$W[q] = \int\limits_{t_1}^{t_2} L\left(q^i(t), \dot{q}^j(t)\right) dt. \tag{13}$$

The form above is equivalent to $\frac{\partial L}{\partial t} = 0$ which implies the energy conservation law $G(q^i, \dot{q}^j) = E$, where

$$G(q^i, \dot{q}^j) = \dot{q}^i \frac{\partial L}{\partial \dot{q}^i} - L \tag{14}$$

is the energy function, and E is the energy constant.

In order to bring the action (13) to a form to which the Routh formalism may be applied, a transformation of the parameter: $t \to \tau$ is performed, defined as $t = \theta(\tau)$, where $\theta'(\tau) \neq 0$, and θ is a meanwhile unknown function. The action (13) transforms then into

$$\mathcal{W}[\theta, x^i] = \int\limits_{\tau_1}^{\tau_2} \Lambda\left(x^i(\tau), \theta'(\tau), x'^j(\tau)\right) d\tau, \tag{15}$$

where

$$\Lambda\left(x^i(\tau), \theta'(\tau), x'^j(\tau)\right) \;=\; L\left(x^i(\tau), \frac{x'^j(\tau)}{\theta'(\tau)}\right)\theta'(\tau), \tag{16}$$

and

$$x^i(\tau) \;:=\; q^i\left(\theta(\tau)\right), \tag{17}$$

$$x'^i(\tau) \;:=\; \dot{q}^i\left(\theta(\tau)\right)\theta'(\tau). \tag{18}$$

The new Lagrangian Λ determines a system of $n+1$ degrees of freedom described by $n+1$ independent variables (θ, x^i) being functions of a parameter τ. (Notation like $x' = \frac{dx}{d\tau}$ *etc* is applied here).

The Lagrangian Λ is a homogeneous function of degree one in the variables (θ', x'^i). The appropriate variational principle determines thus only n independent differential equations of motion regardless of the fact that the system is described by $n+1$ dynamical variables. The Lagrangian Λ does not explicitly depend on θ. Therefore, this variable plays here the same role as q^0 did in the case of the Lagrangian $\mathcal{L}$ discussed before. Equation (2) reads now

$$\begin{aligned} p_0 &= \frac{\partial \Lambda}{\partial \theta'} \\ &= L\left(x^i(\tau), \frac{x'^j(\tau)}{\theta'(\tau)}\right) - \frac{x'^k(\tau)}{\theta'(\tau)}\frac{\partial L}{\partial \dot{q}^k}\left(x^i(\tau), \frac{x'^j(\tau)}{\theta'(\tau)}\right) = -G\left(x^i(\tau), \frac{x'^j(\tau)}{\theta'(\tau)}\right). \end{aligned} \tag{19}$$

Thus $\mathcal{P}_0 = -G(x^i, \frac{x'^j}{\theta'})$, and $p_0 = -E$. Therefore, we have to solve the equation

$$G\left(x^i(\tau), \frac{x'^j(\tau)}{\theta'}\right) = E \tag{20}$$

with respect to θ', prior to starting with the Routh formalism.

Writing the solution as

$$\theta'(\tau) = \phi_E\left(x^i(\tau), x'^j(\tau)\right), \tag{21}$$

we are prepared to transform Λ to a corresponding Routh function which is denoted now by L_E,

$$\begin{aligned} L_E(x^i, x'^j) &= \Lambda\left(x^i, \phi_E(x^j, x'^k), x'^l\right) - p_0\,\phi_E(x^i, x'^j) \\ &= \left[L\left(x^i, \frac{x'^j}{\phi_E(x^k, x'^l)}\right) + E\right]\phi_E(x^r, x'^s) \qquad (22) \\ &= x'^i\left[\frac{\partial L}{\partial \dot{q}^i}\left(x^k, \frac{x'^l}{\phi_E(x^r, x'^s)}\right)\right]. \qquad (23) \end{aligned}$$

The Lagrangian L_E, for the first time derived by Jacobi, describes a reduced dynamical system which resulted from eliminating the information about the time evolution from the original system with the Lagrangian L. In other words, the variables $q^i(t)$ which enter L, after the corresponding equations of motion are solved, describe motions of the system in $\mathbb{Q}^n$ which are curves in $\mathbb{Q}^n$ parametrized by the Newtonian time t. On the other hand, the variables x^i that enter L_E describe trajectories (i.e. spatial paths) of the system; these trajectories are only *loci* of points in $\mathbb{Q}^n$. As far the computations that determine the form of the Lagrangian L_E are concerned, the expression (22) is, in my opinion, more suitable for practical computations than the usually quoted expression (23). It is worthwhile to note that the original Lagrangian L provides information about the form of its energy function G, whereas this piece of information is lost from the reduced Lagrangian L_E; from Eq. (14) it follows that its "energy" function identically vanishes, i.e. *no energy – no time evolution*.

One can show, cf. [2], that objects introduced in this section have the following properties.

1. The function ϕ_E is homogeneous of degree one in the variables x'^i, which means that the relation (21) is covariant with respect to reparametrizations $\tau \to \tau'$.
2. This in turn implies that also the Jacobi Lagrangian L_E is a homogeneous function of degree one in the variables x'^i.
3. The rank of the Hesse matrix of L_E is equal to $n-1$.

Points 2 and 3 mean that the Lagrange equations

$$\frac{\delta L_E}{\delta x^i} := \frac{\partial L_E}{\partial x^i} - \frac{d}{d\tau}\left(\frac{\partial L_E}{\partial x'^i}\right) = 0, \qquad (24)$$

together with appropriate initial conditions, can only determine trajectories in $\mathbb{Q}^n$ described by equations of the form

$$F_K(q^1, \ldots, q^n) = 0, \text{ where } K = 1, \ldots, n-1, \qquad (25)$$

or, usually under obvious additional assumptions, of the form $q^K = q^K(q^n)$.

To determine the complete motion $q^i = q^i(t)$ defined by the original Lagrangian L, one has to add to the $n-1$ equations taken out from (24) the equation

$$G\left(q^i(t), \dot{q}^j(t)\right) = E. \qquad (26)$$

Thus, to determine the complete motion, one needs the triple (L_E, G, E). The pair $\left(q^i(t), t\right)$ geometrically represents a world line in the space of states $\mathbb{Q}^n \times \mathbb{R}$ in which the unit taken along the real axis $\mathbb{R}$ is equal to the unit of the Newtonian time t.

Remark. Equation (26) could as well be replaced by the equivalent equation

$$\phi_E\left(q^i(t), \dot{q}^j(t)\right) = 1.$$

5. The inverse Jacobi problem

Let $L_h\left(x^i(\tau), x'^j(\tau)\right)$ be a function homogeneous of degree one in the variables x'^i. A variational principle with L_h taken as the Lagrangian is only determining (non-parametrized) curves in a $\mathbb{Q}^n$. The following questions can be asked here

i. What data should be added to the knowledge of L_h, in order to be able to lift the spatial paths in $\mathbb{Q}^n$ to motions $q^i = q^i(t)$ determined by a Lagrangian $L\left(q^i(t), \dot{q}^j(t)\right)$ such that the given homogeneous Lagrangian L_h is its Jacobi Lagrangian L_E corresponding to E taken as the energy constant?

ii. What is the algorithm that enables us to determine L in terms of an arbitrarily given L_h and what are the necessary additional data that make the solution to the problem unique?

Problem of such a kind was formulated and solved in [2] under the name of *inverse Jacobi problem*. Now I would like to present its solution.

All that said here so far suggests that a good candidate for the additional data would be an arbitrarily assigned function $G(q^i, \dot{q}^j)$ being the hoped-for energy function of the yet unknown Lagrangian L.

After introducing the velocity variable $v^i = \dot{q}^i(t)$, relation (3.2) turns into a partial differential equation

$$v^1 \frac{\partial L}{\partial v^1} + \ldots + v^n \frac{\partial L}{\partial v^n} - L = G \tag{27}$$

for an unknown function $L(v^i)$. In Eq. (27), $G = G(v^i)$ is treated as a given function, and the dependence of L and G on q^i is here suppressed.

Applying the standard methods of integration of partial linear differential equations, a general integral of (27) can be found to have the form

$$L(q^i, v^j) = \sqrt{|\, g_{rs} v^r v^s \,|}\; I\!\left(q^i, \frac{v^j}{\sqrt{|\, g_{kl} v^k v^l \,|}}, \sqrt{|\, g_{pq} v^p v^q \,|}\right) + \Lambda(q^i, v^j), \tag{28}$$

where g_{ij} stands for the metric tensor in the manifold $\mathbb{Q}^n$ (in case such a tensor is absent, one may write down $g_{ij} = \delta_{ij}$), and where

$$I(c^i, \rho) := \int \frac{G(c^i \rho)}{\rho^2}\, d\rho. \tag{29}$$

The function $\Lambda(q^i, v^j)$ in (28) is an arbitrary integration function homogeneous of degree one in the variables v^j. The equation (28) represents a general formula that determines a class of Lagrangians L describing a conservative dynamical system in terms of an *a priori* assigned energy function G of the system and an arbitrary homogeneous Lagrangian Λ.

To solve the problem, we have to remove the arbitrariness of Λ by making use of the requirement that the given homogeneous Lagrangian $L_h(x^i, x'^j)$ be the Jacobi Lagrangian corresponding to the Lagrangian L determined by Eq. (28).

In order to be able to use the definition (22) of L_E, we have to find first the function ϕ_E by solving the equation

$$G\left(x^i, \frac{x'^j}{\phi_E}\right) = E. \tag{30}$$

By using the requirement just mentioned, it is a quite simple technical matter to find the function Λ as a functional of L_h, G, and ϕ_E.

Substituting this functional into (28), one obtains the Lagrangian L which solves the problem posed:

$$\begin{aligned} L(q^i, v^j) \quad = \quad & \sqrt{\mid g_{ij} v^i v^j \mid}\left[I\left(q^i, \frac{v^j}{\sqrt{\mid g_{pq} v^p v^q \mid}}, \sqrt{\mid g_{rs} v^r v^s \mid}\right)\right. \\ & \left. - I\left(q^i, \frac{v^j}{\sqrt{\mid g_{pq} v^p v^q \mid}}, \frac{\sqrt{\mid g_{rs} v^r v^s \mid}}{\phi_E(q^m, v^n)}\right)\right] + L_h(q^i, v^j) - E\phi_E(q^i, v^j). \end{aligned} \tag{31}$$

6. Geodesics in a Lorentzian manifold

Let $g_{\alpha\beta} = g_{\alpha\beta}(\xi^\gamma)$, $\alpha, \beta = 0, 1, ..., n$, be a Lorentzian metric in a local coordinate system $\{\xi^\alpha\}$ in a manifold $\mathbb{M}^{n+1}$. The choice of its signature is $+ - \ldots -$. The geodesic lines $\xi^\alpha = \xi^\alpha(t)$ in $\mathbb{M}^{n+1}$, parametrized by an affine parameter t, are defined by the action

$$\mathcal{W} = -\tfrac{1}{2} \int_{\tau_1}^{\tau_2} g_{\alpha\beta}\, u^\alpha\, u^\beta\, dt, \tag{32}$$

where $u^\alpha = \frac{d\xi^\alpha}{dt}$. The action (32) determines geodesics as *loci* of points in an $n+2$-dimensional space $\mathbb{R} \times \mathbb{M}^{n+1}$, where $\mathbb{R}$ is the parameter axis. The space $\mathbb{R} \times \mathbb{M}^{n+1}$ is here, unlike in Newtonian mechanics, defined only locally over a geodesic line being just under consideration. In the case of the action (32), let us denote its "energy" function by $\tilde{G}$. By making use of Eq. (14), the function $\tilde{G}$ can be found in the form

$$\tilde{G}(\xi^\alpha, u^\beta) = -\tfrac{1}{2}\, g_{\alpha\beta}\, u^\alpha u^\beta. \tag{33}$$

If one assigns now to the "energy" constant C the value

$$C = -\tfrac{1}{2}\, \varepsilon\, m^2\, c^2, \tag{34}$$

where $\varepsilon = \pm 1$, and m and c are some constants, then by solving the Euler-Lagrange equations, with $m \neq 0$, for $\varepsilon = 1$ one obtains timelike, and for $\varepsilon = -1$ spacelike geodesics. The assumption $m = 0$ is used here in case one wants to obtain a null geodesic.

Since the Lagrangian in the action (32) does not depend explicitly on t, so it is possible here to perform the Jacobi reduction. To this end, one must first solve

Eq. (20) in which G is replaced by $\tilde{G}$ from Eq. (33), and E by C defined in Eq. (34). Thus the solution (21) takes now the form

$$\theta'(\tau) = \phi_E(x^\alpha, x'^\beta) = \frac{1}{m\,c}\sqrt{\varepsilon\, g_{\alpha\beta}\, x'^\alpha\, x'^\beta}, \tag{35}$$

where $x^\alpha(\tau) = \xi^\alpha\,(\theta(\tau))$ and $x'^\alpha = \frac{dx^\alpha}{d\tau}$. Note that the Jacobi reduction is not possible in the case of null geodesics.

Now with the aid of Eq. (22), the Jacobi Lagrangian L_C corresponding to the Lagrangian $\tilde{L}(\xi^\alpha, x'^\beta)$ of the action (32) can be easily found as

$$L_C(x^\alpha, x'^\beta) = -\varepsilon\, mc\sqrt{\varepsilon\, g_{\alpha\beta}\, x'^\alpha\, x'^\beta}. \tag{36}$$

The Lagrangian $L_C(x^\alpha, x'^\beta)$ is homogeneous of degree one in x'^α, with all the consequences of this fact which were indicated above. Thus, in case one would not like to introduce any additional constraint condition, geodesics can be described analytically only by equations of e.g. the form $x^i = x^i(x^0)$, $i = 1, \ldots, n$. This means that the geodesics are *loci* of points in the manifold $\mathbb{M}^{n+1}$, i.e., in this manifold, they are world lines in the terminology used in the theory of relativity.

7. A Newtonian dynamical system

Let us consider in $\mathbb{Q}^n$ a system defined by the Lagrangian

$$\mathcal{L} = \tfrac{1}{2}\, e_{ij}\, v^i v^j + \tfrac{e}{c} A_k v^k - V, \tag{37}$$

where the notation is a standard one; $i, j = 1, ..., n$. It is assumed that the kinetic energy tensor e_{ij}, as well as the potentials A_k and V are functions of only the coordinates q^i in $\mathbb{Q}^n$, and they do not depend explicitly on the time t. The system satisfies then the energy conservation principle

$$\mathcal{G} = \tfrac{1}{2}\, e_{ij}\, v^i v^j + V = \mathcal{E}. \tag{38}$$

If one wishes to apply to Eqs. (37) and (38) the Jacobi reduction procedure described in Sec. 4, one has to solve with respect to θ' first the algebraic equation

$$\frac{e_{ij}\, x'^i x'^j}{2\,\theta'} + V = \mathcal{E} \tag{39}$$

corresponding in the present case to Eq. (20), and next to substitute into the equation which corresponds now to Eq. (22) the solution of Eq. (39), which is

$$\theta' = \phi_{\mathcal{E}}(x^k, x'^l) = \frac{e_{ij}\, x'^i x'^j}{2\,(\mathcal{E} - V)}. \tag{40}$$

The outcome of all the operations just described is the Jacobi Lagrangian

$$\mathcal{L}_{\mathcal{E}} = \sqrt{2\,(\mathcal{E} - V)\, e_{ij}\, x'^i x'^j} + \tfrac{e}{c} A_i x'^i \tag{41}$$

of the system defined by the Lagrangian (37); for the notation cf. Eqs. (17-18). The Lagrangian (41) determines spatial paths in $\mathbb{Q}^n$, whereas the Lagrangian (37) is defining motions in $\mathbb{Q}^n$ which could be looked upon as world lines in $\mathbb{R} \times \mathbb{Q}^n$, where $\mathbb{R}$ is the Newtonian time axis.

8. Geodesics in a stationary space-time

Let

$$\mathcal{S} = -\varepsilon\, mc \int_{(q^0)_1}^{(q^0)_2} \sqrt{\varepsilon \left(g_{00} + 2\, g_{0k} \frac{dq^k}{dq^0} + g_{kl} \frac{dq^k}{dq^0} \frac{dq^l}{dq^0} \right)}\, dq^0 \tag{42}$$

be an action for geodesics, $q^k = q^k(q^0)$, $k = 1, ..., n$, in a space $\mathbb{M}^{n+1}$ with coordinates q^α $(\alpha = 0, 1, ..., n)$. It is assumed that all $g_{\alpha\beta}$ do not depend explicitly on q^0. The minus sign is standing here to assure a principle of the least action, as well as the positive definiteness of energy.

After replacing q^0 by $t = \frac{q^0}{c}$, the Lagrangian corresponding to (42) can be expressed as

$$L(q^k, v^l) = -mc^2\, \varepsilon \sqrt{\varepsilon \left(g_{00} + 2\, g_{0k} \frac{v^k}{c} + g_{kl} \frac{v^k v^l}{c^2} \right)}, \tag{43}$$

where $v^k = \dot{q}^k(t)$. Stationarity of $\mathbb{M}^{n+1}$ implies the conservation of energy

$$G = mc^2 \frac{g_{00} + g_{0k}\, v^k/c}{\sqrt{\varepsilon\, (g_{00} + 2\, g_{0k}\, v^k/c + g_{kl}\, (v^k v^l)/c^2)}} = E. \tag{44}$$

Let us apply now the Jacobi procedure presented in Sec. 4 to the Lagrangian (43) taken together with its energy function (44). It is a fairly straightforward matter to show that in case the function G is given by the expression (44), the algebraic equation (20) on θ' has a unique solution of the form (21) in which for the function ϕ_E one must take

$$\phi_E = -\frac{g_{0k}}{c\, g_{00}}\, x'^k + \frac{E}{c\sqrt{g_{00}}} \sqrt{\frac{\gamma_{ij}\, x'^i x'^j}{E^2/c^2 - m^2\, c^2\, \varepsilon\, g_{00}}}, \tag{45}$$

where

$$\gamma_{ij} = -\left(g_{ij} - \frac{g_{0i} g_{0j}}{g_{00}} \right) \tag{46}$$

is the so-called space metric tensor, cf. [1], and where the notation introduced in Eqs. (17) and (18) applies.

With the aid of Eq. (22), the corresponding Jacobi Lagrangian L_E can now be easily found as

$$L_E(x^i, x'^j) = \sqrt{\left(\frac{E^2}{c^2\, g_{00}} - m^2\, c^2\, \varepsilon \right) \gamma_{ij}\, x'^i x'^j} - \frac{E\, g_{0k}}{c\, g_{00}}\, x'^k. \tag{47}$$

The Lagrangian (47) determines spatial trajectories in $\mathbb{M}^n$ being a section of $\mathbb{M}^{n+1}$ with the hypersurface $x^0 = \text{const}$.

Let us note that due to the equivalence principle, the mass parameter m that enters the Lagrangian (43), unlike the parameter ε, does not appear in the Euler-Lagrange equations of motion which follow from this Lagrangian. These equations of motion admit however a whole class of solutions for which

$$g_{00} + 2\, g_{0k} \frac{v^k}{c} + g_{kl} \frac{v^k v^l}{c^2} = 0, \tag{48}$$

i.e. for which $L = 0$. Thus these motions, represented by null geodesics in $\mathbb{M}^{n+1}$, are not determined by an action principle based on the action (42). Solving the energy conservation law (44) with respect to the square root of the expression standing on the l.h. side of Eq. (48), one can see that the square root tends to zero for $m \to 0$ for the values of ε and E kept fixed and different from zero. Therefore, vanishing of the expression in (48) can be considered to be equivalent to $\lim m = 0$.[1]

From Eq. (47) it follows that L_E is a meaningful Jacobi Lagrangian also for $m = 0$. Therefore, despite the fact that the action (42) does not work for null geodesics in $\mathbb{M}^{n+1}$, the corresponding Jacobi action based on the Lagrangian L_E, given by Eq. (47) for $m = 0$, defines spatial paths in $\mathbb{M}^n$ of such geodesics in $\mathbb{M}^{n+1}$. In that case L_E is the Lagrangian of Fermat's principle for stationary space-times. This principle, thought in a different theoretical framework, was already discussed e.g. in [4].

For $m \neq 0$, the action principle based on the Lagrangian L_E given by (47) can be considered as being a generalization of Fermat's principle for non-null geodesics in a stationary space time $\mathbb{M}^{n+1}$. A Lagrangian of this kind, but only for static space times, was discussed in [6], which unfortunately is a paper with many logical and technical errors. In neither, however, of the papers just mentioned, the true dynamical origin of the principles discussed there was revealed.

Let us finally observe that one can identify the manifolds $\mathbb{M}^n$ and $\mathbb{Q}^n$ of Sec. 7. This is implied by the fact that after making the identifications

$$e_{ij} = \gamma_{ij}; \tag{49}$$

$$eA_i = -\frac{g_{0k}}{g_{00}}\, E; \tag{50}$$

$$V = \mathcal{E} + \tfrac{1}{2}\, m^2 c^2 \varepsilon - \frac{E^2}{2\, g_{00}\, c^2}; \tag{51}$$

and fixing the values of m, e, $\mathcal{E}$, E, one can uniquely express the quantities $g_{\alpha\beta}$ through e_{ij}, A_k, V, or *vice versa*.

The identification of the spaces $\mathbb{Q}^n$ and $\mathbb{M}^n$ and the relations (49)-(51) demonstrate not only the equivalence of the two dynamics defined, correspondingly, by L_E and $\mathcal{L}_\mathcal{E}$, but they also reveal the existence of maps leading from e.g. the dynamics determined by $\mathcal{L}$ to that by L or the other way round, in accordance with the diagrams

$$\mathcal{L},\, (\mathcal{L},\, \mathcal{E}) \xrightarrow{\text{Jacobi}} \mathcal{L}_\mathcal{E},\; \underset{(G,\, E)}{\exists}\; \mathcal{L}_\mathcal{E} + (G,\, E) \xrightarrow[]{\substack{\text{inverse}\\ \text{Jacobi}}} L, \tag{52}$$

and

$$L,\, (L,\, E) \xrightarrow{\text{Jacobi}} L_E,\; \underset{(\mathcal{G},\, \mathcal{E})}{\exists}\; L_E + (\mathcal{G},\, \mathcal{E}) \xrightarrow[]{\substack{\text{inverse}\\ \text{Jacobi}}} \mathcal{L}, \tag{53}$$

where the notation refers to objects that were already discussed in this article.

To demonstrate the way of how diagrams of this kind should be read, let us explain it by taking the diagram (52) as an example. The starting point here is the non-degenerate Lagrangian $\mathcal{L}$ of the form (37) which describes the motion of a system as

[1]Of course, a similar equivalence could have been obtained by putting down $m = 1$ and letting ε tend to zero. The way accepted in the article seems to be a more physical one. It demonstrates that a null geodesic is a limiting case of either timelike or spacelike geodesics which are selected by choosing either one of the two values of the discrete parameter $\varepsilon = \pm 1$, while the mass parameter accepts its values from a continuous interval.

a *locus* of points in the $n+1$-dimensional space $\mathbb{R} \times \mathbb{Q}^n$. The knowledge of $\mathcal{L}$ uniquely determines, by means of Eq. (14), the energy function $\mathcal{G}$ given by (38). A choice that must be made before the Jacobi reduction procedure is started is that of selecting a value of the energy constant $\mathcal{E}$ in Eq. (39). Thus, to start the Jacobi reduction, one has to select a pair $(\mathcal{L}, \mathcal{E})$ in $\mathbb{R} \times \mathbb{Q}^n$. The outcome of the Jacobi procedure is a homogeneous Lagrangian $\mathcal{L}_{\mathcal{E}}$ which is made equivalent to the Lagrangian L_E by means of Eqs. (49)-(51). Now, there exists a pair (G, E), consisting of a function $G(x^i, x'^j)$ and a value of a constant E, such that when the piece of information encoded in the pair is logically added to that encoded in the Lagrangian $\mathcal{L}_{\mathcal{E}}$, one obtains the starting point of an inverse Jacobi procedure that leads us to the target Lagrangian $L(q^k, v^l)$ given by Eq. (43). Of course, for every $\mathcal{L}_{\mathcal{E}}$ there is only one pair (G, E) that allows us to obtain the Lagrangian (43).

9. Geodesics in a stationary space-time in an affine parametrization

The action (42) can be easily transformed to a homogeneous form. This may be achieved by introducing an additional dynamical variable $q^0(\tau)$ as a function of a new parameter τ. Its values are here denoted by x^0, i.e. $x^0 = q^0(\tau)$; and the remaining dynamical variables are then transformed into $x^k = x^k(\tau) := q^k\left(q^0(\tau)\right)$. After changing the integration variable $q^0 \to \tau$, $dq^0 = x'^0\, d\tau$, the integral (42) takes the form

$$\mathcal{S}_h = -\varepsilon\, mc \int_{\tau_1}^{\tau_2} \sqrt{\varepsilon\, g_{\alpha\beta}\, x'^\alpha x'^\beta}\, d\tau, \tag{54}$$

where all $g_{\alpha\beta}$ in the integrand do not depend explicitly on x^0. The two actions, given respectively by (42) and (54), determine the same *loci* of points in the space $\mathbb{M}^{n+1}$ provided the function $x^0(\tau)$ in (54) is not a fixed one, but it is treated like any other dynamical variable during the variational procedure. This property of the action (54) is due to the fact that its Lagrangian is a homogeneous function of degree one in the velocities x'^α. Thus the two dynamics, defined by the actions (42) and (54) respectively, are mutually equivalent.[2]

There are various methods of transforming the action (54) into another one which would give us solutions in a form of parametrized curves in a suitably defined configuration space $\mathbb{Q}$ or, differently speaking, solutions which are world lines in locally defined spaces $\mathbb{R} \times \mathbb{Q}$, where $\mathbb{R}$ is the parameter axis. All such methods amount to adding new information to that encoded in the action (54). For instance, one can substitute for the function $x^0(\tau)$ in (54) any given, monotonous function $x^0 = \tilde{x}^0(\tau)$. This turns the action (54) into a non-homogeneous one which determines parametrized curves $x^k = \xi^k(\tau)$, where $\xi^k(\tau) := x^k\left(\tilde{x}^0(\tau)\right)$, in the space $\mathbb{Q} = \mathbb{M}^n$.

In this section, it is the inverse Jacobi procedure that is to be used. The Lagrangian of the homogeneous action (54) is

$$L_h(x^k, x'^\beta) = -\varepsilon\, mc \sqrt{\varepsilon\, g_{\alpha\beta}\, x'^\alpha x'^\beta}, \tag{55}$$

[2]In some old classical texts on differential geometry the action (54) is referred to as describing geodesics in an *arbitrary* parametrization. This phrase is, however, slightly confusing, for it is used to mean *in a not yet specified parametrization.*

i.e. it is formally equal to the Jacobi Lagrangian given by Eq. (36), but now it does not depend explicitly on x^0. The formal equality may indicate that the procedure we are going to use is, in a sense, inverse to the reduction procedure discussed in Sec. 6. Thus, seemingly, to trace the inverse procedure, it would be sufficient to read the equations of Sec. 6 in a reverse order from Eq. (36) to (32). Although one could in this manner obtain a piece of helpful information, yet the inverse Jacobi method is more than this. It is in a way a procedure of lifting dynamics of a certain type from a configuration space $\mathbb{Q}$ to dynamics of a different type in the space $\mathbb{R} \times \mathbb{Q}$, where $\mathbb{R}$ is the axis of a meanwhile unknown parameter $t = \theta(\tau)$, which is implicitly introduced by a choice of an "energy" function. In principle, the choice of such a function is fairly arbitrary. In practice, however, this choice may be a guess based on the Jacobi reduction method applied to certain hoped-for target Lagrangians of the inverse procedure.

In the present case, the starting homogeneous Lagrangian is L_h given by Eq. (55), and the space $\mathbb{Q} = \mathbb{M}^{n+1}$. In accordance with Sec. 6, the "energy" function is chosen to be

$$\tilde{G}(\xi^k, u^\beta) := -\tfrac{1}{2}\, g_{\alpha\beta}\, u^\alpha u^\beta, \tag{56}$$

where $g_{\alpha\beta} = g_{\alpha\beta}\,(\xi^k)$, $\xi^\alpha(t) = x^\alpha\left((\theta^{-1}(t)\right)$, $u^\alpha = \frac{d\xi^\alpha}{dt}$, and $t = \theta(\tau)$, for $\theta'(\tau) \neq 0$, is a new parameter. Also the choice of the "energy" constant C is, in principle, arbitrary. In order, however, to obtain a desired target Lagrangian, it is chosen, in accordance with Eq. (34), as $C = -\frac{1}{2}\,\varepsilon\, m^2 c^2$. The next step consists in solving Eq. (20) adapted to the present notation. Its solution is presented in Eq. (35). After changing in the expression for $\phi_C(x^k, x'^\beta)$, given by Eq. (35), the names of the variables from (x^k, x'^β) to (ξ^k, u^β), we substitute this expression and that for the function $\tilde{G}(\xi^k, u^\beta)$ given by Eq. (56) into Eq. (31), to obtain the target Lagrangian in the form

$$\tilde{L}(\xi^k, u^\beta) = -\tfrac{1}{2}\, g_{\alpha\beta}\, u^\alpha u^\beta. \tag{57}$$

i.e. a Lagrangian of the same form as that in the action (32), but now the Lagrangian (57) does not depend explicitly on ξ^0. The parameter t introduced by the choice of the "energy" function (56) is an affine one. The Lagrangian (57) determines world lines in $\mathbb{R} \times \mathbb{M}^{n+1}$, where $\mathbb{R}$ stands for the t axis.

The Lagrangian (57) depends neither on t nor on ξ^0. Its independence of t gave rise to the possibility of the Jacobi reduction procedure which was performed in Sec. 6, and here it would restore the starting Lagrangian L_h. Although the Lagrangian (57) is independent of ξ^0, it does depend on $u^0 = \frac{d\xi^0}{dt}$, so ξ^0 is a typical cyclic variable, and the existence of such a variable enables us to apply the Routh reduction procedure to the Lagrangian $\tilde{L}$ as well.

For the sake of this procedure, we replace in the Lagrangian $\tilde{L}$ the names of the variables u^α with v^α and write down Eq. (57) in a way that explicitly exposes the dependence of $\tilde{L}$ on the variable v^0:

$$\tilde{L}(\xi^k, v^\beta) = -\tfrac{1}{2}\,\left(g_{00}(v^0)^2 + 2\, g_{0k} v^0 v^k + g_{kl} v^k v^l\right), \tag{58}$$

In order to eliminate from the Lagrangian (58) the variables (x^0, v^0), we have to compute the quantity $\mathcal{P}_0$ defined in Eq. (2) for the case considered now. We have

$$\frac{\partial \tilde{L}}{\partial v^0} = \mathcal{P}_0 := -g_{00} v^0 - g_{0k} v^k = p_0, \tag{59}$$

and the solution for v^0 of the last equation above is

$$v^0 = \phi\left(p_0, \xi^i(t), v^j(t)\right) := -\frac{p_0}{g_{00}} - \frac{g_{0k} v^k}{g_{00}}. \tag{60}$$

The Lagrangian (4) equals the Routh function $\mathcal{R}\left(\xi^i, \phi(p_0, \xi^k, v^l), v^j, t\right)$ and takes now the form

$$\mathcal{L}_{p_0}(\xi^k, v^l) = \tfrac{1}{2}\,\gamma_{kl} v^k v^l + p_0\,\frac{g_{ok}}{g_{00}}\,v^k + \frac{p_0^2}{2\,m g_{00}}. \tag{61}$$

The Lagrangian above is of the same type as that defined by Eq. (37). Therefore, we can identify the configuration space of dynamics defined by the Lagrangian (61) with the configuration space $\mathbb{R} \times \mathbb{Q}^n$ of the dynamics discussed in Sec. 7. Comparing in the two Lagrangians, $\mathcal{L}$ and $\mathcal{L}_{p_0}$ respectively, the coefficients at the same powers of v^k, we obtain

$$\begin{aligned} e_{ij} &= \gamma_{ij}; && (62)\\ eA_i &= p_0 c\,\frac{g_{0k}}{g_{00}} && (63)\\ V &= -\frac{{p_0}^2}{2\,g_{00}} + \text{const.} && (64) \end{aligned}$$

Comparing next the two sets of relations, represented respectively by Eqs. (49)-(51) and by Eqs. (62)-(64), we see that Eqs. (49) and (62) are identical, and Eq. (50) and (63) can be made identical by assuming that $p_0\,c = -E$. Then Eq. (64) turns into

$$V = -\frac{E^2}{2\,g_{00}\,c^2} + \text{const}, \tag{65}$$

which demonstrates that the two dynamics, defined respectively by $\mathcal{L}$ and $\mathcal{L}_{p_0}$, are fully equivalent.

The content of this section is a generalization of a result by Eisenhart [5] who has shown that the trajectories of a general holonomic conservative system of n degrees of freedom in classical dynamics can be put into correspondence with geodesics of a suitable Riemannian manifold $\mathcal{S}$, where $\dim \mathcal{S} = n+1$. In [5], however, no use of methods of analytical dynamics was made, in particular of those concerning cyclic variables, but instead only tedious transformations of the underlying ODE were performed.

Another reason which enables us to consider the result just obtained as a more general one than that of Eisenhart is that it permits one to prolong the sequence of mappings shown in the diagram (52) by a new sequence presented in the following diagram

$$L \xrightarrow{\substack{\text{reparame-}\\ \text{trization}}} L_h,\ \underset{(\tilde{G},\,C)}{\exists}\ L_h + (\tilde{G},\,C) \xrightarrow{\substack{\text{inverse}\\ \text{Jacobi}}} \tilde{L},\ (\tilde{L}, p_0) \xrightarrow{\text{Routh}} \mathcal{L}_{p_0} \equiv \mathcal{L}. \tag{66}$$

The complete sequence, made by joining the sequences (66) and (52) one after the other, forms a closed loop. In an analogous way, with the help of the algorithms presented in this article, one can also prove the validity of the following sequence of mappings

$$\mathcal{L},\ \underset{(\mathcal{P}_0,\,p_0)}{\exists}\ \mathcal{L} + (\mathcal{P}_0,\,p_0) \xrightarrow{\substack{\text{inverse}\\ \text{Routh}}} \tilde{L},\ (\tilde{L},\,C) \xrightarrow{\text{Jacobi}} L_h \equiv L. \tag{67}$$

The composition {(53), (67)} of the corresponding sequences forms again a loop of mappings which passes exactly through the same dynamics as the previous loop, but the other way round.

Thus the two loops, {(52), (66)} and {(53), (67)} taken together, define a commuting diagram of mappings between all the dynamics discussed above. In terms of pairs consisting of Lagrangians and spaces of states[3] of corresponding dynamics, the diagram may be shown as

$$\begin{array}{ccc} (\tilde{L},\, \mathbb{R} \times \mathbb{M}^{n+1}) & \longleftrightarrow & (L_h\,,\mathbb{M}^{n+1}) \equiv (L,\, \mathbb{M}^{n+1}) \\ \updownarrow & & \updownarrow \\ (\mathcal{L},\, \mathbb{R} \times \mathbb{Q}^n) & \longleftrightarrow & (\mathcal{L}_{\mathcal{E}}\,,\, \mathbb{Q}^n) \equiv (L_E,\, \mathbb{M}^n). \end{array} \tag{68}$$

In this diagram the names of the procedures which labelled the corresponding mapping arrows, as well as other details concerning the definitions of mappings are suppressed, but they may be easily recovered by means of the diagrams (52), (53), (66), and (67).

It is rather remarkable that the seemingly arbitrary constant in Eq. (64) can be easily determined. This follows from the fact that the two dynamics, $(\tilde{L},\, \mathbb{R} \times \mathbb{M}^{n+1})$ and $(\mathcal{L},\, \mathbb{R} \times \mathbb{Q}^n)$, are invariant under translations of respective parameters t in the two configuration spaces. And this fact was not exploited yet. The invariance induces in the space $\mathbb{R} \times \mathbb{M}^{n+1}$ the conservation law: $\tilde{G}(\xi^k, u^\beta) = -\frac{1}{2}\,\varepsilon\, m^2 c^2$, in accordance with Eqs. (64) and (34). In order to project this conservation law on the space $\mathbb{R}\times \mathbb{Q}^n$, one must replace in it the variables u^α with v^α, eliminate from it the variable v^0, and make use of Eq. (60), replacing $p_0\, c$ by $-E$. After all this is done, one obtains

$$-\tfrac{1}{2}\,\gamma_{ij} v^i v^j + \frac{E^2}{2\, g_{00}\, c^2} = \tfrac{1}{2}\,\varepsilon\, m^2 c^2. \tag{69}$$

On the other hand, the same invariance of the dynamics $(\mathcal{L},\, \mathbb{R} \times \mathbb{Q}^n)$ induces the conservation law (38). Upon making use of the relation (62), and eliminating from Eq. (38) the potential V by means of Eq. (65), one transforms Eq. (38) into

$$\tfrac{1}{2}\,\gamma_{ij} v^i v^j - \frac{E^2}{2 g_{00} c^2} = \mathcal{E} - \mathrm{const}\,. \tag{70}$$

Eliminating now the kinetic term from Eqs. (69) and (70), one finds that

$$\mathrm{const} = \mathcal{E} + \frac{1}{2}\,\varepsilon\, m^2 c^2, \tag{71}$$

which shows that the relations (51) and (65) agree with each other.

The last result indicates that the mappings from the diagram (68) preserve various features of the three types of geodesics, labelled by the values of ε and m.

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[3] In the terminology introduced by Synge [7], a *space of states* of a dynamical system is the space in which the motions determined by the dynamics are represented by curves being *loci* of points.

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